Von Neumann Models with Infinitely Many Goods and Processes[1]

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I. Introduction

There is now available a huge literature concerning the existence proof of von Neumann growth equilibrium. The original proof due to Neumann [11] utilized the Brower fixed-point theorem. Kemeny–Morgenstern–Thompson [7] is based on a game theoretic approach. Howe's proof [3] is concise, depending on a theorem due to Tucker [13], while Gale [2] uses the Minkowski–Farkas lemma. In all these models, consumers' freedom to choose commodities is ruled out.

In a von Neumann model as introduced by Morishima [8, 9], consumers' choice is taken into accunt and in so doing, labor input coefficients and consumption baskets are considered in an explicit way. His proof is based on a fixed-point theorem.

On the other hand, it has been pointed out by Hülsmann–Steinmetz [4] that in a von Neumann model *a la* Gale [1] with infinite processes, the existence of equilibrium is not always guaranteed and the Gale's proof in [1] is incorrect, even for the finite case. This difficulty is related to the lack of the Kuhn–Tucker constraint qualification in nonlinear programming.

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(1) Thanks are due to Prof. Morishima who pointed out to me the topic in this essay.

So, let us deal with in this essay Morishima–type von Neumann models in which there can be infinitely many goods and processes. In this respect our result below seems new and our proof is performed through a simple maximization problem, though in our model capitalists' consumption is assumed away. In secsion II, we prove a mathematical proposition, making use of a theorem by Hurwicz [5]. Section III is devoted to raising as examples some kinds of von Neumann models for which the existence of equilibrium is an easy outcome of our mathematical proposition in sectipn II.

II. Mathematical Proposition

Let us first state our notation.

- X, C : locally convex topological vector space.
- K_x, K_c: closed, convex and pointed cones with nonempty interiors in X and C respectively.
- M : linear operator on K_x to C.
- R^n : Euclidean space of dimension n.
- R^n_+ : nonnegative orthant of R^n.
- $*$: symbol to indicate conjugate spaces or conjugate cones.
- L : linear operator on K_x to R^1.
- u : continuous concave function on K_c to R^1.

We specify X, C as locally convex in order to assure the existence of non-trivial linear functionals. An order $\geqq$ is defined in X, C, X^* and C^* using cones K_x, K_c, K^*_x and K^*_c respecfively. A symbol $>$ is used in the same meaning as in Hurwicz [5, p. 66]. A symbol 0 means both the zero vector and the zero functional in various spaces, but the meaning is clear from the context.

Now consider the following problem:

(P) Maximize $u(c)$ subject to $x \in K_x, c \in K_c$,

$Mx - c \geqq 0$, and

$1 - Lx \geqq 0$.

To investigate this problem, we make the following assumptions.

A1. There exists an x in K_c such that $Mx > 0$ (i. e., Mx is an interior point of K_c).

A2. (i) If $c \in K_c$ and $c > 0$, then $u(c) > u(0)$. (ii) If $u(c) > u(0)$ for a $c \in K_c$. then $u(kc) > u(c)$ for a scalar $k > 1$.

A3. The problem (P) has a solution (x^0, c^0).

A solution pair guaranteed by A3 may not be unique, then fix one of them as (x^0, c^0). Now we prove

Proposition. Given the assumptions A1–A3, there exists a triplet (x^0, c^0, y^0) such that $x^0 \in K_x, c^0 \in K_c, y^0 \in K_c^*$,

$$Mx^0 \geqq c^0, \tag{1}$$

$$y^0 Mx^0 = y^0 c^0, \tag{2}$$

$$y^0 Mx \leqq Lx \text{ for any } x \in K_x, \tag{3}$$

$$y^0 Mx^0 = Lx^0, \tag{4}$$

$$y^0 c^0 > 0, \tag{5}$$

$$Lx^0 = 1, \text{ and} \tag{6}$$

$$u(c^0) \geqq u(c) \text{ for any } c \in K_c \text{ such that } y^0 c \leqq y^0 c^0, \tag{7}$$

where $y^0 Mx$ means $y^0(Mx)$.

Proof. Use is made of THEOREM V. 3. 1. by Hurwicz [5, p. 91]. We first give the following list which shows the correspondence between Hurwicz's symbols and ours.

script $X \cdots X \times C$,

script $Y \cdots R^1$,

script $Z \cdots C \times R^1$,

$P_z \cdots K_c \times R^1_+$,

$X \cdots K_x \times K_c$,

$f \cdots (x, c) \in K_x \times K_c \longrightarrow u(c) \in R^1$,

$g \cdots (x, c) \in K_x \times K_c \longrightarrow (Mx - c, 1 - Lx) \in C \times R^1$.

It is not difficult to see that the requirements of Hurwicz's theorem are all satisfied. We know that the problem (P) has a solution (x^0, c^0) such that $Mx^0 \geqq c^0$ and $Lx^0 \leqq 1$. Thus, applying THEOREM V. 3.1, we have $z^0 \in K^*_c$ and $k^0 \in R^1_+$ such that

$$u(c) + z^0(Mx - c) + k^0(1 - Lx) \tag{8}$$

$$\leqq u(c^0) + z^0(Mx^0 - c^0) + k^0(1 - Lx^0) \tag{9}$$

$$\leqq u(c^0) + z(Mx^0 - c^0) + k(1 - Lx^0) \tag{10}$$

for any $x \in K_x$, $c \in K_c$, $z \in K^*_c$ and $k \in R^1_+$.

From the second inequality, we have

$$z^0(Mx^0 - c^0) = 0, \text{ and} \tag{11}$$

$$k^0(1 - Lx^0) = 0. \tag{12}$$

For, if one of them were negative, then the value of the third term, (10), could be made smaller than that of (9). Next rewrite the terms (8) and (9) into

$$u(c) + (z^0 Mx - k^0 Lx) + (k^0 - z^0 c) \tag{8'}$$

$$\leqq u(c^0) + (z^0 Mx^0 - k^0 Lx^0) + (k^0 - z^0 c^0). \tag{9'}$$

From this, we have

$$z^0 Mx - k^0 Lx \leqq 0 \text{ for any } x \in K_x. \tag{13}$$

Otherwise, the value of (8′) could be made greater than that of (9′), a contradiction. Also we have

$$z^0 Mx^0 - k^0 Lx^0 = 0 \tag{14}$$

for the same reason. From (9′) it follows $k^0 = z^0 c^0$.

Now we have to show that $z^0 c^0 > 0$. Assume the contrary, $z^0 c^0 = 0$. Then, by substituting kc^0 and x^0 for c and x in (8), we obtain from (8) and (9), $u(kc^0) \leqq u(c^0)$ for all $k \in R^1_+$. A contradiction to the assumption A2 when $k > 1$. Thus,

$$z^0 c^0 > 0. \tag{15}$$

So, $k^0 > 0$. Define $y^0 = z^0/k^0 \in K^*_c$. Then equations or inequalities (11), (13), (14), (15) and (12) become (2), (3), (4), (5) and (6) respectively. (7) can be shown as follows. Substitute x^0 for x in (8), and we obtain from (8) and (9), $u(c) - z^0 c \leqq u(c^0) - z^0 c^0$ for all $c \in K_c$. This implies $u(c^0) - u(c) \geqq k^0(y^0 c^0 - y^0 c)$ for all $c \in K_c$. This completes the proof. Q. E. D.

III. Von Neumann Models

In this section, we raise some von Neumann models for which the existence of equilibrium is guaranteed by our proposition.

(i) Interpretation for the Finite-Dimensional Case

Let us first consider a von Neumann model with a finite number of goods and processes. That is, we suppose in our economy there are m goods and n processes. Let B be the output coefficient matrix (m by n) and A the input coefficient matrix (m by n). Each process is described by the two corresponding columns of A and B. Denote by g the natural rate of growth in labor power and define $M \equiv B - (1+g)A$, a mapping from R^n_+ to R^m. L is the labor input coefficient vector. The function u is the utility function which is assumed identical among workers. Then in the model under consideration, K_x is R^n_+, showing the space of activity level vectors, while K_c is R^m_+, meaning the space of commodities. The assumption A1 in the preceding section means that our economy is productive enough to produce a surplus in each good while maintaining economic growth at the rate g. A2 implies that there are commodities, called consumption goods, which yield utility and that there exists no bliss point. If $L>0$, then the set of feasible vectors to the problem (P) is compact and the assumption A3 is automatically satisfied. We assume also that there are two classes, one the capitalist class and the other the working class. Capitalists do not consume, while workers do not save. Workers can, however, choose consumption goods to maximize their utility level. Now our proposition insures the existence of growth equilibrium in such a von Neumann model. The equation (2) is the expression of the rule of free goods ([10, p. 80]), regarding y^0 as an equilibrium price vector (labor as the numeraire). (3) means that under the price vector y^0 no process can realize a rate of profit which is greater than the natural rate g. (4) is the rule of profitability ([10, p. 80]). (5) means the value of the consumption basket of a worker is positive, hence, of course, the value of the total consumption by the working class is positive. Needless to say, the value of the total output is positive, which was first required in equilibrium by Kemeny–Morgenstern–Thompson [7]. (6) implies the employment of a worker, regarding one man-per period as the unit of labor power. Then (7) means that workers choose consumption goods so that they maximize their utility level under the budget constraint, since $y^0c^0(=Lx^0=1)$ is the wages paid to a worker (note labor is the numeraire).

(ii) Countable Goods and Processes

An interesting model may be one in which goods and processes are both countable. This case can be thought of as a limiting case of the preceding finite one. The matrix M is now $\infty\times\infty$. L is $(1,1,\ldots)$, i.e., each process uses labor and is normalized so as to make its labor input unity. Some difficulties arise, especially with the assumption A3. In order for the problem (P) to have a solution, it is sufficient that the set of feasible *commodity* vectors, $F\equiv\{c\,|\,c\in K_c,\ Mx\geqq c$ for some $x\in K_x$ such that $Lx\leqq 1\}$ is compact. So let us consider the case in which X and C are both real l^2, denoted by rl^2, i. e., the set of sequences $\{x\,|\,x=(x_1,\ x_2,\ldots),\ \sum_{i=1}^{\infty}|x_i|^2<\infty,\ x_i\in R^1\}$. K_x and K_c are the set $rl^2_+\equiv\{x\,|\,x\in rl^2, x_i\geqq 0$ for all $i\}$. We assume the matrix M satisfies

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|M_{ij}|^2<\infty, \tag{16}$$

where M_{ij} is the (i, j) entry of M. Then, it is known that M is a completely continuous (or compact) operator (See Riesz–Nagy [12, pp. 195–197]). Since the set $\{x\,|\,x\in rl^2_+,\ Lx\leqq 1\}$ is weakly compact in X, the set F becomes compact. Topologies of X and C are, of course, introduced by the usual norm of l^2 as a Banach space. It is noted here that the assumption AI is hard to satisfy in such a model. But it is possible to replace AI by the following (See Hurwicz–Uzawa [6, p. 104]):

AI′. For any $y\in K_c^*$ such that $y\neq 0$, there exists an $x\in K_x$ such that $yMx>0$.

Now we can apply our proposition above to insure the existence of equilibrium. This model can be extended from l^2 to Banach spaces having a countable basis.

The inequality (16) is, however, quite restrictive, almost implying the finiteness of the number of processes. Is there any wayout, which is meaningful from an economic point of view? One solution is like this. We suppose first that the number of *consumption* goods is finite, let us say, m, and that other non-consumption goods add to no utility, whatever combinations of them are offered to a consumer. X and C are $R^\infty\equiv\{x\,|\,x=(x_1, x_2, \ldots),\ x_i\in R^1\}$, while K_x and K_c are $R^\infty_+\equiv\{x\,|\,x\in R^\infty,\ x_i\geqq 0$ for all $i\}$. The first m rows of M and the first m elements of $c\in K_c$ express amounts of *consumption* goods. Let $E \equiv \{d \mid d \in R^m_+$, d is the first m elements of $c \in K_c\}$. Next assume that elements of M are bounded uniformly both from above and from below, i. e., there exists a positive scalar k such that $|M_{ij}| < k$ for all i, j. L is again $(1, 1, \ldots) \in R^\infty$. Then, the intersection $E \cap F$ is bounded in R^m_+, where F is the set defined above. Now we add one more assumption that $E \cap F$ is closed. Then $E \cap F$ is compact, the assumption A3 being satisfied. One may assume A3 directly for von Neumann models with infinitely many goods and processes.

(iii) Countable Goods and a Continuum of Processes.

Let I be the closed interval $[0, 1]$ on R^1. X is $L^1(I)$, the set of Lebesgue measurable real functions $x(t)$ defined almost everywhere on I such that $|x(t)|$ is Lebesgue integrable on I. K_x is $\{x(t) \mid x(t) \geqq 0$ at defined points on $I\}$. C is R^∞, while K_c is R^∞_+. We suppose again that there are a finite number of *consumption* goods, say, m goods. Let $M_i(t)$, $i = 1, 2 \ldots$, be the set of uniformly bounded (from above and below) real functions on I. That is, there exists a scalar k such that $|M_i(t)| < k$ for all i and $t \in I$. The suffix i indicates the i–th commodity, while a parameter value t on I indicates a corresponding process. The mapping M is defined in a dimension–wise way as $\int_0^1 M_i(t)x(t)dt$ for the i–th commodity, i. e., the i–th dimension of C. Lx is defined as $\int_0^1 x(t)dt$. Then again by making a similar assumption on $E \cap F$ we can apply our proposition. This last example is relevant when smooth substitution is possible between inputs of processes.

REFERENCES

[1] Gale, D.: "The Closed Linear Model of Production," in *Annals of Mathematical Studies*, No. 38. Princeton: Princeton University Press, 1956.

[2] ——; "Comment," *Econometrica*, 40 (1972), 391-392.

[3] Howe, C. W.: "An Alternative Proof of the Existence of General Equilibrium in a von Neumann Model," *Econometrica*, 28 (1960), 635–639.

[4] Hülsmann, J., and V. Steinmetz: "A Note on the Non-existence of Optimal Price Vectors in the General Balanced-Growth Model of Gale," *Econometrica*, 40 (1972), 387–389.

[5] Hurwicz, L.: "Programming in Linear Spaces," in *Studies in Linear and Nonlinear Programming*. Stanford: Stanford University Press, 1958.

[6] Hurwicz, L., and H. Uzawa: "A Note on the Lagrangian Saddle-Points," in *Studies in Linear and Nonlinear Programming*, ibid.

[7] Kemeny, J. G., O. Morgenstern, and G. L. Thompson: "A Generalization of the von Neumann Model of an Expanding Economy," *Econometrica*, 24 (1956), 115–135.

[8] Morishima, M.: "Economic Expansion and the Interest Rate in Generalized von Neumann Models," *Econometrica*, 28 (1960), 352–363.

[9] ——: *Equilibrium. Stability and Growth*. Oxford: Oxford University Press, 1964.

[10] ——: *Theory of Economic Growth*. Oxford: Oxford University Press, 1969.

[11] Neumann, J. von: "A Model of General Economic Equilibrium," *Review of Economic Studies*, 13 (1945—6), 1—9.

[12] Riesz, F., and B. Sz.-Nagy: *Functional Analysis*. New York: Ungar, 1955.

[13] Tucker, A. W.: "Dual Systems of Homogeneous Linear Relations," in *Annals of Mathematical Studies*, ibid. in [1].